

Absolute Extensors and Binary Monads

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Received: 4 September 2014 / Accepted: 7 March 2016 / Published online: 15 March 2016
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Abstract Let \mathbb{F} be a monad in the category *Comp* of compact Hausdorff spaces and continuous maps. An abstract convexity was constructed by Radul for each \mathbb{F} -algebra of the monad \mathbb{F} in the category *Comp*. It was proved that if the convexity of the monad \mathbb{F} with some additional properties is binary then \mathbb{F} has good topological properties, in particular, FX is an absolute extensor in the class of 0-dimensional spaces for each openly generated compactum X . We show in this paper that binarity is also a necessary condition.

Keywords Monad · Convexity · Binarity · Absolute extensor in the class of 0-dimensional compacta

Mathematics Subject Classification (2010) 18B30 · 18C15 · 52A01 · 54B30

1

The notion of convexity considered in this paper is considerably broader than the classical one; specifically, it is not restricted to the context of linear spaces. Such convexities appeared in the process of studying different structures like partially ordered sets, semilattices, lattices, superextensions etc. We base our approach on the notion of topological convexity of [21] where the general convexity theory is covered from axioms to applications in different areas.

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In [13], a certain categorical connection between topological-algebraic structures and convexities was found. Many topological constructions are functorial; they are defined not only for spaces but for maps as well. Certain functors in categories of topological spaces and continuous maps have also natural algebraical structures. Such structures can be described by the notion of monad (or triple) structure in the sense of S.Eilenberg and J.Moore and their corresponding category of Eilenberg-Moore algebras [3].

Many classical constructions lead to monads: hyperspaces, spaces of probability measures, superextensions etc. (See for example the survey [12] or the book [18]). A number of authors consider certain convexity structures to investigate monads. The linear convexity corresponds to the probability measure monad [17]. Certain non-linear convexities correspond to the superextension monad [7], the inclusion hyperspace monad [16] and the idempotent measure monad [22]. A convexity structure on each \mathbb{F} -algebra for any monad \mathbb{F} in the category of compact Hausdorff spaces and continuous maps was introduced in [13]. These convexities are preserved by \mathbb{F} -algebra morphisms. Let us remark that our general construction of convexities includes the above mentioned convexities for probability measures, superextension, hyperspaces of inclusion and idempotent measures.

Let us consider two classical theorems about topological properties of the probability measure functor P and the superextension functor λ .

Theorem 1.1 (Ditor, Haydon [1]) *For a compactum X the space of probability measures PX is an absolute retract if and only if X is an openly generated compactum with weight $\leq \omega_1$.*

Let us remark that in [1] the above theorem was formulated for Dugundji spaces but it is known that each openly generated compactum of weight $\leq \omega_1$ is a Dugundji space [20, Th. 5].

Theorem 1.2 (Ivanov [8]) *For a compactum X the superextension λX is an absolute retract if and only if X is an openly generated continuum.*

The notion of absolute retract (shortly AR) coincides with the notion of absolute extensor (shortly AE) in the class of compacta (see for example [11]). The notion of absolute extensor in the class of 0-dimensional compacta (shortly AE(0)) is considerably broader than the notion of absolute extensor. For example, the Cantor cube D^τ is AE(0) but certainly not AE. However, these notions are equivalent in the realm of linear convexity. Each (linearly) convex compactum is AE if and only if it is AE(0) [5]. Since the compactum PX is convex, we have that PX is not even AE(0) for any compactum X with weight $\geq \omega_2$. But we have that λX is AE(0) for each openly generated compactum X [9]. This difference between properties of functors one can explain in terms of properties of the corresponding convexities. The convexity generated by the superextension has the binarity property, which the linear convexity does not have. It was shown in [13] that each monad which generates a binary convexity has good topological properties. More precisely, Theorem 3.3 of [13] states that for each binary L-monad \mathbb{F} which weakly preserves preimages (we also assume that the functorial part F of the monad \mathbb{F} preserves monomorphisms, intersections and one-point spaces) and for each compactum X such that FX is an openly generated compactum, each map $f : FX \rightarrow Y$ with \mathbb{F} -convex fibers is 0-soft provided f is open. If we put $Y = \{*\}$ we obtain that for each binary L-monad \mathbb{F} which weakly preserves preimages FX is AE(0) provided FX to be an openly generated compactum. Let us remark that it is enough for X

to be openly generated compactum in case when the functor F is open (i.e. preserves open maps).

The main aim of this paper is to show that the binarity property is also necessary: if FPD^{ω_2} is $\text{AE}(0)$ (for a monad \mathbb{F} with some additional natural properties) then the corresponding convexity is binary (where PD^{ω_2} is the space of probability measures of the Cantor cube of the weight ω_2). Let us remark that PD^{ω_2} is an openly generated compactum (P is an open functor [2]) which is not $\text{AE}(0)$ by Theorem 1.1.

The paper is organized as follows. In Section 2 we recall necessary definitions and the construction of an abstract convexity for a monad of [13]. In Section 3 we obtain a certain sufficient condition for a monad to be binary and in Section 4 we obtain the main result.

2

By *Comp* we denote the category of compact Hausdorff spaces (compacta) and continuous maps. Let X be a compactum. We denote by $C(X)$ the Banach space of all continuous functions $\varphi : X \rightarrow \mathbb{R}$ with the usual sup-norm: $\|\varphi\| = \sup\{|\varphi(x)| \mid x \in X\}$. In what follows, all spaces and maps are assumed to be in *Comp* except for \mathbb{R} and maps in sets $C(X)$ with X compact Hausdorff.

The family \mathcal{C} of closed subsets of X is called a *convexity* on X if \mathcal{C} is stable under intersection and contains X and the empty set. The elements of \mathcal{C} are called \mathcal{C} -convex. Let us remark that we modify the definition of [21]. We consider only closed convex sets. A convexity in the sense of [21] can be obtained as the collection of all unions of up-directed families of sets in \mathcal{C} .

We need some definitions concerning monads and algebras (see [10] for more details). We define them only for the category *Comp*.

A *monad* $\mathbb{T} = (T, \eta, \mu)$ in the category *Comp* consists of an endofunctor $T : \text{Comp} \rightarrow \text{Comp}$ and natural transformations $\eta : \text{Id}_{\text{Comp}} \rightarrow T$ (unity), $\mu : T^2 \rightarrow T$ (multiplication) satisfying the relations $\mu \circ T\eta = \mu \circ \eta T = \text{Id}_T$ and $\mu \circ \mu T = \mu \circ T\mu$. (By Id_{Comp} we denote the identity functor on the category *Comp* and T^2 is the superposition $T \circ T$ of T .)

Let $\mathbb{T} = (T, \eta, \mu)$ be a monad in the category *Comp*. A pair (X, ξ) , where $\xi : TX \rightarrow X$ is continuous map, is called a \mathbb{T} -*algebra* if $\xi \circ \eta X = \text{id}_X$ and $\xi \circ \mu X = \xi \circ T\xi$.

Let (X, ξ) be an \mathbb{F} -algebra for a monad $\mathbb{F} = (F, \eta, \mu)$ and A be a non-empty closed subset of X . Denote by f_A the quotient map $f_A : X \rightarrow X/A$ (the classes of equivalence are the one-point sets $\{x\}$ for $x \in X \setminus A$ and the set A) and put $a = f_A(A)$. Denote $A^+ = (Ff_A)^{-1}(\eta(X/A)(a))$. (Let us remark that the set $A^+ \subset FX$ could be define for each closed subset A of any compactum X , without additional requirement on X to be an \mathbb{F} -algebra). It is easy to check that $A^+ \subset B^+$ for each pair of closed subsets of X with $A \subset B$. Define the \mathbb{F} -convex hull $C_{\mathbb{F}}(A)$ of A by $C_{\mathbb{F}}(A) = \xi(A^+)$. Put additionally $C_{\mathbb{F}}(\emptyset) = \emptyset$. We define the family $\mathcal{C}_{\mathbb{F}}(X, \xi) = \{A \subset X \mid A \text{ is closed and } C_{\mathbb{F}}(A) = A\}$. The elements of the family $\mathcal{C}_{\mathbb{F}}(X, \xi)$ are called \mathbb{F} -convex. It was shown in [13] that the family $\mathcal{C}_{\mathbb{F}}(X, \xi)$ forms a convexity on X .

3

We recall and investigate the property of binarity for a monad. Let \mathcal{S} be a family of subsets of a set X . We say that \mathcal{S} is *linked* if the intersection of each pair of elements of \mathcal{S} is non-empty. \mathcal{S} is called *binary* if the intersection of every linked subsystem is non-empty. We call

a monad \mathbb{F} binary if $\mathcal{C}_{\mathbb{F}}(X, \xi)$ is binary for each \mathbb{F} -algebra (X, ξ) . We shall find a certain sufficient condition for a monad $\mathbb{F} = (F, \eta, \mu)$ to be binary.

Proposition 3.1 *Let \mathcal{S} be a closed under intersection family of subsets of a set X such that each linked subfamily consisting of 3 elements has non-empty intersection. Then each finite linked subfamily of \mathcal{S} has non-empty intersection.*

Proof Let \mathcal{C} be a finite linked subfamily of \mathcal{S} . Let us use induction on cardinality of \mathcal{C} , starting from 3. For $|\mathcal{C}| = 3$ the statement is trivial. Assume that our statement is true for each subfamily \mathcal{C} with $|\mathcal{C}| \leq n \geq 3$. Consider a linked subfamily $\mathcal{C} = \{C_0, C_1, \dots, C_n\}$ and define a family $\mathcal{C}' = \{C_0 \cap C_1, \dots, C_0 \cap C_n\}$. Since each 3-element subfamily of \mathcal{C} has non-empty intersection, we have that the family \mathcal{C}' is linked, hence it has non-empty intersection by induction assumption. But $\cap \mathcal{C} = \cap \mathcal{C}'$ and the proposition is proved. \square

Lemma 3.2 *Let $f : X \rightarrow Y$ be a continuous map. Then $Ff(B^+) \subset (f(B))^+$ for each closed subset $B \subset X$.*

Proof Let $\chi_B : X \rightarrow X/B$ be the quotient map. Put $b = \chi_B(B)$. Let $\chi_{f(B)} : Y \rightarrow Y/f(B)$ be the quotient map and put $b' = \chi_{f(B)}(f(B))$. There exists a unique map $f' : X/B \rightarrow Y/f(B)$ such that $f' \circ \chi_B = \chi_{f(B)} \circ f$. Let us remark that the map f' is continuous and $f'(b) = b'$. Consider any $v \in Ff(B^+)$. Then there exists $\mu \in B^+$ such that $Ff(\mu) = v$. Since $\mu \in B^+$, we have $F(\chi_B)(\mu) = \eta(X/B)(b)$. Since η is a natural transformation, we have $Ff' \circ F(\chi_B)(\mu) = \eta(Y/f(B))(b')$. On the other hand $Ff' \circ F(\chi_B) = F(\chi_{f(B)}) \circ Ff$, hence $F(\chi_{f(B)})(v) = \eta(Y/f(B))(b')$ and $v \in (f(B))^+$. \square

We say that a functor F preserves one-point preimages iff for each continuous map $h : X \rightarrow Y$ and each $y \in Y$ with $h^{-1}(y) = \{x\}$ we have $(Fh)^{-1}(\eta Y(y)) = \{\eta X(x)\}$.

Lemma 3.3 *Let $f : X \rightarrow Y$ be a continuous map and a functor F preserves one-point preimages. Then we have $(Ff)^{-1}(A^+) = (f^{-1}(A))^+$ for each closed $A \subset Y$ such that $f^{-1}(A) \neq \emptyset$.*

Proof Let $\chi_A : Y \rightarrow Y/A$ be the quotient map. Put $a = \chi_A(A)$. Let $\chi_{f^{-1}(A)} : X \rightarrow X/f^{-1}(A)$ be the quotient map and put $a' = \chi_{f^{-1}(A)}(f^{-1}(A))$. There exists a unique map $f' : X/f^{-1}(A) \rightarrow Y/A$ such that $f' \circ \chi_{f^{-1}(A)} = \chi_A \circ f$. Let us remark that the map f' is continuous and $(f')^{-1}(a) = \{a'\}$.

Consider any $v \in (f^{-1}(A))^+$. Then we have $F(\chi_{f^{-1}(A)})(v) = \eta(X/f^{-1}(A))(a')$. Since η is a natural transformation, we have $Ff' \circ F(\chi_{f^{-1}(A)})(v) = \eta(Y/A)(a)$. Since $Ff' \circ F(\chi_{f^{-1}(A)}) = F(\chi_A) \circ Ff$, we have $Ff(v) \in A^+$, hence $v \in (Ff)^{-1}(A^+)$.

Now let $v \in (Ff)^{-1}(A^+)$. Then we have $Ff' \circ F(\chi_{f^{-1}(A)})(v) = F(\chi_A) \circ Ff(v) = \eta(Y/A)(a)$. Since F preserves one-point preimages and $(f')^{-1}(a) = \{a'\}$, we have $F(\chi_{f^{-1}(A)})(v) = \eta(X/f^{-1}(A))(a')$, hence $v \in (f^{-1}(A))^+$. \square

It is easy to see that $\{a\}^+ = \{\eta Y(a)\}$ for each $a \in Y$. Hence we obtain the following corollary considering one-point set in the previous lemma.

Corollary 3.4 *Let $f : X \rightarrow Y$ be a continuous map and a functor F preserves one-point preimages. Then we have $(Ff)^{-1}(\eta Y(a)) = (f^{-1}(a))^+$ for each $a \in Y$ such that $f^{-1}(a) \neq \emptyset$.*

A compactum X is called 0-dimensional if X has a base consisting of open-and-closed sets. For each finite open cover $\{U_1, \dots, U_n\}$ of a 0-dimensional compactum X there exists a finite open cover $\{V_1, \dots, V_n\}$ such that $V_i \subset U_i$ for each $i \in \{1, \dots, n\}$ and $V_i \cap V_j = \emptyset$ provided $i \neq j$. Let us remark that sets V_i are closed as well. (See [4] for more information about 0-dimensional spaces).

Let us remark that the epimorphisms in the category *Comp* are exactly the continuous surjections.

Theorem 3.5 *Let $\mathbb{F} = (F, \eta, \mu)$ be a monad such that F preserves epimorphisms and one-point preimages. If there exist a compactum X and a linked system $\{A_1, A_2, A_3\}$ of closed subsets of X such that $\bigcap_{i=1}^3 A_i = \emptyset$ and $\bigcap_{i=1}^3 A_i^+ \neq \emptyset$, then \mathbb{F} is binary.*

Proof Consider any \mathbb{F} -algebra (Y, ξ) and a linked family $\mathcal{B} \subset \mathcal{C}_{\mathbb{F}}(Y, \xi)$. We will show that $\bigcap \mathcal{B} \neq \emptyset$. Since Y is compact, it is enough to consider $\mathcal{B} = \{B_1, B_2, B_3\}$ by Proposition 3.1. We can choose a 0-dimensional compactum Z and a continuous surjective map $h : Z \rightarrow X$. Then the family $\{h^{-1}(A_i) \mid i \in \{1, 2, 3\}\}$ is a linked family with empty intersection. Since the compactum Z is 0-dimensional, we can write $Z = \bigcup_{i=1}^3 V_i$ where the sets V_i are closed, pairwise disjoint, $V_i \supset \bigcap_{j \in \{1, 2, 3\} \setminus \{i\}} h^{-1}(A_j)$ and $V_i \cap h^{-1}(A_i) = \emptyset$.

We can choose a continuous map $f : Z \rightarrow Y$ such that $f(V_i) \subset \bigcap_{j \in \{1, 2, 3\} \setminus \{i\}} B_j$ for each $i \in \{1, 2, 3\}$. Since $\bigcap_{i=1}^3 A_i^+ \neq \emptyset$ and F preserves epimorphisms, there exists $\alpha \in \bigcap_{i=1}^3 (Fh)^{-1}(A_i^+)$. We have $\bigcap_{i=1}^3 (Fh)^{-1}(A_i^+) = \bigcap_{i=1}^3 (h^{-1}(A_i))^+$ by Lemma 3.3. Since $h^{-1}(A_i) \subset \bigcup_{j \in \{1, 2, 3\} \setminus \{i\}} V_j$, we have $f(h^{-1}(A_i)) \subset \bigcup_{j \in \{1, 2, 3\} \setminus \{i\}} f(V_j) \subset \bigcup_{j \in \{1, 2, 3\} \setminus \{i\}} \bigcap_{k \in \{1, 2, 3\} \setminus \{j\}} B_k \subset B_i$ for each $i \in \{1, 2, 3\}$. Then we obtain that $Ff(\alpha) \in Ff(\bigcap_{i=1}^3 (h^{-1}(A_i))^+) \subset \bigcap_{i=1}^3 Ff((h^{-1}(A_i))^+) \subset$ (by Lemma 3.2) $\subset \bigcap_{i=1}^3 (f(h^{-1}(A_i)))^+ \subset \bigcap_{i=1}^3 B_i^+$, hence $\bigcap_{i=1}^3 B_i^+ \neq \emptyset$. Then we have $\bigcap_{i=1}^3 B_i = \bigcap_{i=1}^3 \xi(B_i^+) \neq \emptyset$. \square

4

The class of L -monads was introduced in [15] under another name and it contains many well-known monads in *Comp* like superextension, hyperspace, probability measures etc (see also [14] for a similar notion). For $\varphi \in C(X)$ by $\max \varphi$ ($\min \varphi$) we denote $\max_{x \in X} \varphi(x)$ ($\min_{x \in X} \varphi(x)$) and by π_φ or $\pi(\varphi)$ we denote the corresponding projection $\pi_\varphi : \prod_{\psi \in C(X)} [\min \psi, \max \psi] \rightarrow [\min \varphi, \max \varphi]$. It was shown in [15] that for each L -monad $\mathbb{F} = (F, \eta, \mu)$ one can consider FX as a subset of the product $\prod_{\varphi \in C(X)} [\min \varphi, \max \varphi]$, moreover, we have $\pi_\varphi \circ \eta X = \varphi$, $\pi_\varphi \circ \mu X = \pi(\pi_\varphi)$ for all $\varphi \in C(X)$ and $\pi_\psi \circ Ff = \pi_\psi \circ f$ for all $\psi \in C(Y)$, $f : X \rightarrow Y$. We could consider these properties of L -monads as a definition [15].

What follows we shall assume that each monad is an L -monad with some fixed inclusion $FX \subset \prod_{\varphi \in C(X)} [\min \varphi, \max \varphi]$. We will use a simpler notation $\nu(\varphi) = \pi_\varphi(\nu)$ for $\nu \in FX$ and $\varphi \in C(X)$. It is easy to check that for each closed subset $A \subset X$ we have $A^+ = \{\nu \in FX \mid \nu(\varphi) = c \text{ for each } \varphi \in C(X) \text{ such that } \varphi|_A \equiv c \text{ for some } c \in \mathbb{R}\}$. Then the family

$\mathcal{O} = \{O(\varphi_1, \dots, \varphi_k; c_1, \dots, c_k; \varepsilon) \mid k \in \mathbb{N}, \varphi_i \in C(X), c_i \in \mathbb{R} \text{ such that } \varphi_i|_A \equiv c_i \text{ and } \varepsilon > 0\}$ forms a base of neighborhoods of A^+ , where $O(\varphi_1, \dots, \varphi_k; c_1, \dots, c_k; \varepsilon) = \{v \in FX \mid |\nu(\varphi_i) - c_i| < \varepsilon \text{ for each } i \in \{1, \dots, k\}\}$. Indeed, consider a closed subset $B \subset FX$ such that $B \cap A^+ = \emptyset$. Then for each point $\beta \in B$ there exist $\varphi_\beta \in C(X)$, $\varepsilon_\beta > 0$ and $c_\beta \in \mathbb{R}$ such that $\varphi_\beta|_A \equiv c_\beta$ and $|\beta(\varphi_\beta) - c_\beta| > \varepsilon_\beta$. Put $V_\beta = \{v \in FX \mid |\nu(\varphi_\beta) - c_\beta| > \varepsilon_\beta\}$. Then V_β is an open subset of FX such that $\beta \in V_\beta \subset FX \setminus A^+$. Since FX is compact, we can choose $\beta_1, \dots, \beta_k \in B$ such that $\bigcup_{i \in \{1, \dots, k\}} V_{\beta_i} \supset B$. Put $\varepsilon = \min\{\varepsilon_{\beta_i} \mid i \in \{1, \dots, k\}\}$. Then $A^+ \subset O(\varphi_{\beta_1}, \dots, \varphi_{\beta_k}; c_{\beta_1}, \dots, c_{\beta_k}; \varepsilon) \subset FX \setminus B$.

Lemma 4.1 *Let $\mathbb{F} = (F, \eta, \mu)$ be an L -monad which is not binary and such that F preserves epimorphisms and one-point preimages. Let X be a compactum, C be a non-empty closed subset of X and U be an open neighborhood of C . Let $\{(A_\alpha; B_\alpha) \mid \alpha \in \Lambda\}$ be a family of pairs of closed subsets of X such that $\emptyset \neq A_\alpha \cap B_\alpha \subset X \setminus U$ for each $\alpha \in \Lambda$. Then there exists an open subset $V \subset FX$ such that $C^+ \subset V$ and $A_\alpha^+ \cap B_\alpha^+ \subset FX \setminus V$ for each $\alpha \in \Lambda$.*

Proof Choose a 0-dimensional compactum Y and a surjective map $f : Y \rightarrow X$. Since Y is 0-dimensional, we can choose for each $\alpha \in \Lambda$ non-empty closed, pairwise disjoint subsets Y_1, Y_2^α and Y_3^α of Y (Y_1 does not depend on α) such that $Y \setminus f^{-1}(C) \supset Y_1 \supset f^{-1}(X \setminus U)$, $Y_2^\alpha \supset f^{-1}(A_\alpha) \setminus Y_1$, $Y_3^\alpha \supset f^{-1}(B_\alpha) \setminus Y_1$ and $Y_1 \cup Y_2^\alpha \cup Y_3^\alpha = Y$. For $\alpha \in \Lambda$ consider the continuous map $h_\alpha : Y \rightarrow \{1, 2, 3\}$ defined as follows: $h_\alpha(Y_1) = \{1\}$, $h_\alpha(Y_2^\alpha) = \{2\}$ and $h_\alpha(Y_3^\alpha) = \{3\}$ (we consider $\{1, 2, 3\}$ with the discrete topology).

Since \mathbb{F} is not binary, we have $\{1, 2\}^+ \cap \{2, 3\}^+ \cap \{1, 3\}^+ = \emptyset$ by Theorem 3.5. Since $\{1, 2\}^+ \cap \{1, 3\}^+$ is a closed subset in $F\{1, 2, 3\}$, there exist $\varepsilon > 0$, $c_1, \dots, c_k \in \mathbb{R}$ and functions $\psi_1, \dots, \psi_k \in C(\{1, 2, 3\})$ such that $\psi_i(2) = \psi_i(3) = c_i$ and for each $v \in \{1, 2\}^+ \cap \{1, 3\}^+$ there exists $i \in \{1, \dots, k\}$ such that $|\nu(\psi_i) - c_i| \geq \varepsilon$.

Let us remark that for each $\alpha, \beta \in \Lambda$ and $i \in \{1, \dots, k\}$ we have $\psi_i \circ h_\alpha = \psi_i \circ h_\beta$. Hence for each $i \in \{1, \dots, k\}$ there exists $\varphi_i \in C(Y)$ such that $\varphi_i = \psi_i \circ h_\alpha$ for each $\alpha \in \Lambda$. Put $D_i = \{\mu \in FY \mid |\mu(\varphi_i) - c_i| \geq \varepsilon\}$ and $V = FX \setminus (\bigcup_{i=1}^k Ff(D_i))$.

Consider any $\alpha \in \Lambda$ and $v \in A_\alpha^+ \cap B_\alpha^+$. Since F preserves epimorphisms, there exists $\mu \in FY$ such that $Ff(\mu) = v$. Then we have $\mu \in (f^{-1}(A_\alpha))^+ \cap (f^{-1}(B_\alpha))^+$ by Lemma 3.3. Since $f^{-1}(A_\alpha) \subset Y_1 \cup Y_2^\alpha$ and $f^{-1}(B_\alpha) \subset Y_1 \cup Y_3^\alpha$, we obtain $\mu \in (Y_1 \cup Y_2^\alpha)^+ \cap (Y_1 \cup Y_3^\alpha)^+$. Hence $Fh_\alpha(\mu) \in \{1, 2\}^+ \cap \{1, 3\}^+$ by Lemma 3.2. Then there exists $i \in \{1, \dots, k\}$ such that $\varepsilon \leq |Fh_\alpha(\mu)(\psi_i) - c_i| = |\mu(\psi_i \circ h_\alpha) - c_i| = |\mu(\varphi_i) - c_i|$ and $v \in Ff(D_i) \subset FX \setminus V$.

Now, consider any $v \in C^+$ and any $\mu \in FY$ such that $Ff(\mu) = v$. By Lemma 3.3, $\mu \in (f^{-1}(C))^+ \subset (Y_2^\alpha \cup Y_3^\alpha)^+$ for each $\alpha \in \Lambda$. By Lemma 3.2 $Fh_\alpha(\mu) \in \{2, 3\}^+$. Hence $c_i = Fh_\alpha(\mu)(\psi_i) = \mu(\varphi_i)$ for each $i \in \{1, \dots, k\}$ and $v \in FX \setminus (\bigcup_{i=1}^k Ff(D_i)) = V$. The lemma is proved. \square

We will need certain information about inverse systems and spectral methods developed by Shchepin to study topology of non-metrizable compacta (see [19] for more details). Suppose that A is a partially ordered up-directed set. A subset B of A is said to be closed if for each chain C in B we have $\sup C \in B$ whenever $\sup C$ exists in A . An up-directed set A is called τ -complete (where τ is an infinite cardinal) if for every chain B in A with $|B| \leq \tau$ there exists an element $\sup B$ in A . A continuous inverse system $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, A\}$ is called a τ -system if its index set A is τ -complete and if the spaces X_α , $\alpha \in A$, have

weight $\leq \tau$. By $\lim \mathcal{S}$ and by $p_\alpha, \alpha \in A$, we denote the limit space of our system and the natural limit projections from $\lim \mathcal{S}$ to X_α . The system $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, A\}$ with $X = \lim \mathcal{S}$ is called factorizing if for each continuous function $f : X \rightarrow \mathbb{R}$ there exist $\alpha \in A$ and a continuous function $f_\alpha : X_\alpha \rightarrow \mathbb{R}$ such that $f_\alpha \circ p_\alpha = f$. Each τ -system such that all X_α are compacta is factorizing [19, Proposition 1.6].

We will use the following fact, which is a particular case of Shchepin Spectral Theorem [19]. Let $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, A\}$ and $\mathcal{L} = \{Y_\alpha, q_\alpha^\beta, A\}$ be two τ -systems where all X_α and Y_α are compacta and τ is an infinite cardinal. Then for each continuous map $f : X \rightarrow Y$ between the corresponding limit spaces $X = \lim \mathcal{S}$, $Y = \lim \mathcal{L}$ there exists a cofinal closed subset $B \subset A$ and a family $\{f_\beta : X_\beta \rightarrow Y_\beta\}_{\beta \in B}$ of continuous maps such that $f_\beta \circ p_\beta = q_\beta \circ f$ for each $\beta \in B$. Moreover, if f is a homeomorphism, we can assume that all f_β are homeomorphisms.

A functor F is called continuous if it preserves the limits of inverse systems. Let us remark that the probability measure functor P is continuous, preserves weights of infinite compacta and epimorphisms (see [6] for more information about the functor P).

A compactum X is called an absolute extensor in the class of 0-dimensional compacta (shortly AE(0)) if for any 0-dimensional compact space Z , any closed subspace Y of Z and any continuous map $\Phi : Y \rightarrow X$ there exists a continuous map $G : Z \rightarrow X$ such that $G|_Y = \Phi$.

Theorem 4.2 *Let $\mathbb{F} = (F, \eta, \mu)$ be an L -monad such that F is continuous, preserves weights of infinite compacta, epimorphisms and one-point preimages. If $F(PD^A) \in AE(0)$ for a set A with $|A| \geq \omega_2$, then the monad \mathbb{F} is binary.*

Proof Let $F(PD^A) \in AE(0)$ for a set A with $|A| \geq \omega_2$. Firstly we introduce some notations. For any subsets $A'' \subset A' \subset A$ we denote by $\pi_{A''}^{A'} : D^{A'} \rightarrow D^{A''}$ and $\rho_{A''}^{A'} : I^{A'} \rightarrow I^{A''}$ the natural projections. We also put $p_{A''}^{A'} = F(P\pi_{A''}^{A'}) : F(PD^{A'}) \rightarrow F(PD^{A''})$. Let us write $\pi_{A''}$, $\rho_{A''}$ and $p_{A''}$ for $\pi_{A''}^A$, $\rho_{A''}^A$ and $p_{A''}^A$, respectively.

Consider any embedding $i : F(PD^A) \rightarrow I^A$. Put $A_1 = \bigsqcup_{\alpha \in A} N_\alpha$ where each $N_\alpha = \mathbb{N}$. Evidently $|A_1| = |A|$ and $D^{A_1} \cong \prod_{\alpha \in A} D^{N_\alpha}$, thus we shall identify these two spaces. Let $s_\alpha : D^{N_\alpha} \rightarrow I$ be a continuous surjective map. Define a map $k : D^{A_1} \rightarrow I^A$ as follows: $k = \prod_{\alpha \in A} s_\alpha$. Since $F(PD^A) \in AE(0)$, there exists a map $f : D^{A_1} \rightarrow F(PD^A)$ such that $i \circ f(z) = k(z)$ for each $z \in k^{-1}(i(F(PD^A)))$.

Let $\mathcal{A} = \{C \subset A \mid \text{card } C \leq \omega_1\}$ and put $C_1 = \bigsqcup_{\alpha \in C} N_\alpha$ for each $C \in \mathcal{A}$. By $k_C : D^{C_1} \rightarrow I^C$ we denote the map $k = \prod_{\alpha \in C} s_\alpha$. Consider the following three inverse systems $\{I^C, \rho_B^C, \mathcal{A}\}$, $\{D^{C_1}, \pi_{B_1}^{C_1}, \mathcal{A}\}$ and $\{F(PD^C), p_B^C, \mathcal{A}\}$ over the directed set \mathcal{A} . Evidently, the first two systems are ω_1 -systems with the limit spaces I^A and D^{A_1} respectively. Since the functors F and P are continuous and preserve weights of infinite compacta, the third system is an ω_1 -system with the limit space $F(PD^A)$.

Consider the continuous map $i : F(PD^A) \rightarrow I^A$. Consider the subfamily $\mathcal{B}_1 \subset \mathcal{A}$ consisting of all $B \in \mathcal{A}$ for which there exists a continuous map $i_B : F(PD^B) \rightarrow I^B$ such that the diagram

$$\begin{array}{ccc} F(PD^A) & \xrightarrow{p_B} & F(PD^B) \\ \downarrow i & & \downarrow i_B \\ I^A & \xrightarrow{\rho_B} & I^B \end{array}$$

is commutative. Then by Shchepin Theorem discussed before \mathcal{B}_1 is a cofinal closed subfamily of \mathcal{A} . Moreover, considering the homeomorphism $i' : F(PD^A) \rightarrow i'(F(PD^A))$ and the ω_1 -system $\{\rho_C(i'(F(PD^A))), \rho_B^C | \rho_C(i'(F(PD^A))), \mathcal{A}\}$ we can state that all i_B are topological embeddings.

Now consider the continuous map $f : D^{A_1} \rightarrow F(PD^A)$ and the subfamily $\mathcal{B}_2 \subset \mathcal{A}$ consisting of all $B \in \mathcal{A}$ for which there exists a continuous map $f : D^{B_1} \rightarrow F(PD^B)$ such that the diagram

$$\begin{array}{ccc} D^{A_1} & \xrightarrow{\pi_{B_1}} & D^{B_1} \\ \downarrow f & & \downarrow f_B \\ F(PD^A) & \xrightarrow{p_B} & F(PD^B) \end{array}$$

is commutative. As before \mathcal{B}_2 is a cofinal closed subfamily of \mathcal{A} .

Then the family $\mathcal{B} = \mathcal{B}_1 \cap \mathcal{B}_2$ is cofinal too. Moreover, following either the proof of Lemma 2 from [1] or the proof of Shchepin theorem [19] (both are based on so-called spectral search which makes countable number of steps) we can assume that for each infinite set $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subset B$ and $|A| = |B|$. We note that for each $B \in \mathcal{B}$ the diagram

$$\begin{array}{ccc} F(PD^A) & \xrightarrow{p_B} & F(PD^B) \\ \downarrow i & & \downarrow i_B \\ I^A & \xrightarrow{\rho_B} & I^B \\ \uparrow k & & \uparrow k_B \\ D^{A_1} & \xrightarrow{\pi_{B_1}} & D^{B_1} \\ \downarrow f & & \downarrow f_B \\ F(PD^A) & \xrightarrow{p_B} & F(PD^B) \end{array}$$

is commutative. It is easy to check that we have $i_B \circ f_B(z) = k_B(z)$ for each $z \in k_B^{-1}(i_B(F(PD^A)))$.

Choose a set $B \in \mathcal{B}$ such that $|B| = \omega_1$. Choose any point $a \in A \setminus B$ and countable infinite subset $F \subset B$. Then there exists a countable set $E \in \mathcal{B}$ such that $\{a\} \cup F \subset E$. Then we have $B \setminus E \neq \emptyset \neq E \setminus B$ and $B \cap E = C \in \mathcal{B}$ where C is an infinite countable set. Put $T = \{(\mu, \nu) \in F(PD^B) \times F(PD^E) | p_C^B(\mu) = p_C^E(\nu)\}$. For each $(\mu, \nu) \in T$ we have $\rho_C^B \circ i_B(\mu) = i_C \circ p_C^B(\mu) = i_C \circ p_C^E(\nu) = \rho_C^E \circ i_E(\nu)$. So, there is a (uniquely determined) element z of $I^{B \cup E}$ such that $\rho_B^{B \cup E}(z) = i_B(\mu)$ and $\rho_E^{B \cup E}(z) = i_E(\nu)$. We can define a map $j : T \rightarrow I^{B \cup E}$ by the formula $j(\mu, \nu) = z$. It is easy to check that the map j is an embedding.

Define a continuous map $\sigma : D^{B_1 \cup E_1} \rightarrow F(PD^B) \times F(PD^E)$ by the formula $\sigma = (f_B \circ \pi_{B_1}^{B_1 \cup E_1}, f_E \circ \pi_{E_1}^{B_1 \cup E_1})$. Then for each $t \in D^{B_1 \cup E_1}$ we have $p_C^B \circ f_B \circ \pi_{B_1}^{B_1 \cup E_1}(t) = f_C \circ \pi_{C_1}^{B_1} \circ \pi_{B_1}^{B_1 \cup E_1}(t) = f_C \circ \pi_{C_1}^{B_1 \cup E_1}(t) = f_C \circ \pi_{C_1}^{E_1} \circ \pi_{E_1}^{B_1 \cup E_1}(t) = p_C^E \circ f_E \circ \pi_{E_1}^{B_1 \cup E_1}(t)$. Hence $\sigma(t) \in T$.

Define a continuous map $p : F(PD^A) \rightarrow T$ by the formula $p = (p_B, p_E)$. We obtain a commutative diagram:

$$\begin{array}{ccc}
 F(PD^A) & \xrightarrow{p} & T \\
 \downarrow i & & \downarrow j \\
 I^A & \xrightarrow{\rho_{B \cup E}} & I^{B \cup E} \\
 \uparrow k & & \uparrow k_{B \cup E} \\
 D^{A_1} & \xrightarrow{\pi_{B_1 \cup E_1}} & D^{B_1 \cup E_1} \\
 \downarrow f & & \downarrow \sigma \\
 F(PD^A) & \xrightarrow{p} & T
 \end{array}$$

It is easy to check that we have $j \circ \sigma(z) = k_{B \cup E}(z)$ for each $z \in k_{B \cup E}^{-1}(j(T))$.

Let us show that p is an open map. Consider any $v \in F(PD^A)$ and $K \subset T$ such that $p(v) \in \text{cl } K$. We shall prove that $v \in \text{cl}(p^{-1}(K))$. Let (d_α) be a net in K such that $\lim d_\alpha = p(v)$. Then the net $(j(d_\alpha))$ converges to $j(p(v))$. Since the last diagram is commutative, we have $\rho_{B \cup E}(i(v)) = j(p(v))$. Choose $t_\alpha \in D^{B_1 \cup E_1}$ such that $k_{B \cup E}(t_\alpha) = j(d_\alpha)$ for each α . Since $D^{B_1 \cup E_1}$ is compact, we can assume that t_α converges to some $t \in D^{B_1 \cup E_1}$. Then $k_{B \cup E}(t) = j(p(v)) = \rho_{B \cup E}(i(v))$ and we can choose $z \in D^{A_1}$ such that $k(z) = i(v)$ and $\pi_{B_1 \cup E_1}(z) = t$. Then $f(z) = i^{-1}(k(z)) = v$. Since the map $\pi_{B_1 \cup E_1}$ is open, we can assume that there exists a net (z_α) which converges to z and $\pi_{B_1 \cup E_1}(z_\alpha) = t_\alpha$ for each α . Put $v_\alpha = f(z_\alpha)$. Then the net v_α converges to v . Since the last diagram is commutative, we have $p(v_\alpha) = \sigma(t_\alpha)$. Since $k_{B \cup E}(t_\alpha) = j(d_\alpha)$, we have $\sigma(t_\alpha) = j^{-1}(k_{B \cup E}(t_\alpha)) = d_\alpha$. Hence $\text{cl}(p^{-1}(K)) \supset \text{cl}\{v_\alpha\} \ni v$ and p is open.

Let us remark that the openness of the map p implies the openness of the map $p_1 : FP(D^{B \cup E}) \rightarrow T$ defined as follows: $p_1 = (p_B^{B \cup E}, p_E^{B \cup E})$.

Choose an accumulation point s_0 of D^C , distinct points $r_0, r_1 \in D^{B \setminus C}$ and $t_0, t_1 \in D^{E \setminus C}$. Define a measure $\lambda \in P(D^{B \cup E}) = P(D^{B \setminus C} \times D^C \times D^{E \setminus C})$ as follows:

$$\lambda = \frac{1}{2}(\delta(r_0, s_0, t_0) + \delta(r_1, s_0, t_1)).$$

Choose disjoint closed neighborhoods U_0, U_1 of r_0, r_1 in $D^{B \setminus C}$ and W_0, W_1 of t_0, t_1 in $D^{E \setminus C}$, and let V be the closed neighborhood of λ in $P(D^{B \cup E})$ defined by

$$V = \left\{ \lambda' \in P(D^{B \cup E}) \mid \lambda'(U_0 \times D^C \times W_0 \cup U_1 \times D^C \times W_1) \geq \frac{1}{2} \right\}.$$

Define the map $\gamma : P(D^{B \cup E}) \rightarrow Q = \{(\mu, \nu) \in PD^B \times PD^E \mid P(\pi_C^B)(\mu) = P(\pi_C^E)(\nu)\}$ by $\gamma = (P(\pi_B^{B \cup E}), P(\pi_E^{B \cup E}))$. Now $\gamma(\lambda) = (\mu, \nu)$, where

$$\mu = \frac{1}{2}(\delta(r_0, s_0) + \delta(r_1, s_0))$$

and

$$\nu = \frac{1}{2}(\delta(s_0, t_0) + \delta(s_0, t_1)).$$

For any point $s \in D^C \setminus \{s_0\}$ put

$$\mu_s = \frac{1}{2}(\delta(r_0, s_0) + \delta(r_1, s))$$

and

$$v_s = \frac{1}{2}(\delta(s, t_0) + \delta(s_0, t_1)).$$

Since s_0 is an accumulation point, we have $(\mu, v) \in \text{cl}\{(\mu_s, v_s) \mid s \in D^C \setminus \{s_0\}\}$. Consider any $\lambda_s \in P(D^{B \cup E})$ for which $\gamma(\lambda_s) = (\mu_s, v_s)$. Since $P(\pi_B^{B \cup E})(\lambda_s) = \frac{1}{2}(\delta(r_0, s_0) + \delta(r_1, s))$, we have $\lambda_s = \frac{1}{2}(\lambda_s^1 + \lambda_s^2)$ where $\lambda_s^1 \in P(\{r_0\} \times \{s_0\} \times D^{E \setminus C})$ and $\lambda_s^2 \in P(\{r_1\} \times \{s\} \times D^{E \setminus C})$. Since $P(\pi_E^{B \cup E})(\lambda_s) = \frac{1}{2}(\delta(s, t_0) + \delta(s_0, t_1))$, we have $\lambda_s^1 \in P(\{r_0\} \times \{s_0\} \times \{t_1\})$ and $\lambda_s^2 \in P(\{r_1\} \times \{s\} \times \{t_0\})$. Thus

$$\lambda_s = \frac{1}{2}(\delta(r_0, s_0, t_1) + \delta(r_1, s, t_0)),$$

which is not in V . Hence $(P(\pi_B^{B \cup E}))^{-1}(\mu_s) \cap (P(\pi_E^{B \cup E}))^{-1}(v_s) \cap V = \emptyset$ for each $s \in D^C \setminus \{s_0\}$.

Put $H_s = (P(\pi_B^{B \cup E}))^{-1}(\mu_s)$ and $G_s = (P(\pi_E^{B \cup E}))^{-1}(v_s)$. Then we have $H_s^+ \cap G_s^+ = p_1^{-1}(\eta P D^B(\mu_s), \eta P D^E(v_s))$ by Corollary 3.4. Since p_1 is open, we have $\eta P D^{B \cup E}(\lambda) \in \text{cl}(\cup\{H_s^+ \cap G_s^+ \mid s \in D^C \setminus \{s_0\}\})$. Hence \mathbb{F} is binary by Lemma 4.1 and the theorem is proved. \square

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